Real Analysis
Tutorials (1st week)

1. (a) Let $E_1 \supseteq E_2 \supseteq \cdots \supseteq E_n \supseteq E_{n+1} \supseteq \cdots$ be a decreasing sequence of measurable sets such that $m(E_1) < \infty$. If $E = \bigcap_{n=1}^{\infty} E_n$, then show that $\lim_{n\to\infty} m(E_n) = m(E)$.

   (b) Let $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n \subseteq E_{n+1} \subseteq \cdots$ be an increasing sequence of measurable sets. If $E = \bigcup_{n=1}^{\infty} E_n$, then show that $\lim_{n\to\infty} m(E_n) = m(E)$.

2. **(Borel-Cantelli Lemma)** Let $\{E_n\}$ be a sequence of measurable sets with $\sum_{k=1}^{\infty} m(E_k) < \infty$. If $E = \{x \in \mathbb{R}^d : x \in E_k$ for infinitely many $k\}$, then prove that $m(E) = 0$.

3. If A, B are subsets of $\mathbb{R}^n$ such that $d(A, B) > 0$, then prove that $m^*(AUB) = m^*(A) + m^*(B)$, where m^* denotes the outer measure.

4. **Invarinace Properties**

   (a) If E is measurable, then prove that $E + h$ is measurable and $m(E + h) = m(E)$.

   (b) If E is measurable and r denotes rotation in $\mathbb{R}^d$ around origin, then prove that $r(E)$ is measurable and $m(r(E)) = m(E)$.

   (c) If $T : \mathbb{R}^d \to \mathbb{R}^d$ is an orthogonal map and E is measurable, then show that $T(E)$ is measurable and $m(T(E)) = m(E)$.

5. If $\delta(x_1, x_2, \cdots, x_d) = (t_1 x_1, \cdots, t_d x_d)$ and if E is measurable, then $\delta(E)$ is measurable and $m(\delta(E)) = |t_1 t_2 \cdots t_d| m(E)$.

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6. If $L : \mathbb{R}^d \to \mathbb{R}^d$ is a linear map and E is measurable, then show that $L(E)$ is measurable and $m(L(E)) = |\det L| m(E)$.

7. (a) If $S = \{(x, y) : ax + by + c = 0\}$ (a line in $\mathbb{R}^2$), then show that $m_2(S) = 0$ where m_2 is a 2-dimensional Lebesgue measure in $\mathbb{R}^2$.
   (b) If $S \subset \mathbb{R}^k \subset \mathbb{R}^n$, $(k < n)$, then prove that $m_n(S) = 0$, where m_n is an n-dimensional Lebsegue measure in $\mathbb{R}^n$.

8. If f is a non-negative measurable function such that $\int_{\mathbb{R}^d} f = 0$, then prove that $f = 0$ a.e. on $\mathbb{R}^d$.

9. Give an example of a sequence $\{f_n\}$ of measurable functions such that $f_n(x)$ converges pointwise to $f(x)$ a.e. but $\int f_n$ does not converge to $\int f$.

10. Define $g(z) = \int_{[0,1]} e^{zlogt} dt$ for $z \in \mathbb{C}$ such that $Re(z) > 0$. Show that g is continuos.

11. Let E_1, E_2 be compact sets in $\mathbb{R}^d$ such that $E_1 \subset E_2$. Let $m(E_1) = a$ and $m(E_2) = b$. Prove that for any c such that $a < c < b$, there exists a compact set E such that $E_1 \subset E \subset E_2$ and $m(E) = c$.

12. Give an example of two measurable sets A and B such that $A + B$ is not measurable.

13. Give an example of two closed sets A and B such that $m(A) = m(B) = 0$ but $m(A + B) > 0$.

14. Let f be an integrable function on $\mathbb{R}^d$.
   (a) If $f_h(x) = f(x + h), h \in \mathbb{R}^d$, then prove that f_h is integrable and $\int f = \int f_h$.
   (b) For $\delta > 0$, $f(\delta x)$ is integrable and $\delta^d \int f(x) = \int f(\delta x)$.
   (c) If $h : \mathbb{R}^d \to \mathbb{R}^d$ is an orthogonal map, then show that $\int f \circ h = \int f$.

15. If f and g are integrable functions, then prove that

$$\int_{\mathbb{R}^d} f(x - y) g(y) = \int_{\mathbb{R}^d} f(y) g(x - y).$$

16. If f is integrable on $(-\pi, \pi]$ and extended periodically to $\mathbb{R}$ with period 2π, then show that $\int_{-\pi}^{\pi} f = \int_I f$ for any interval I of length 2π. in